Phase-space density in heavy-ion collisions revisited

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Abstract. We derive the phase space density of bosons from a general boson interferometry formula. We find that the phase space density is connected with the two-particle and the single-particle density distribution functions. If the boson density is large, the two-particle density distribution function cannot be expressed as a product of two single-particle density distributions. However, if the boson density is so small that the two-particle density distribution function can be expressed as a product of two single-particle density distributions, then Bertsch's formula is recovered. For a Gaussian model, the effects of multi-particles Bose–Einstein correlations on the mean phase space density are studied.

1 Introduction

The principal aim of the study of relativistic heavy-ion collisions is the search for evidence of the state of a quark gluon plasma (QGP) in the early stage of the reactions [1, 2]. A quantity of great interest for the study of the QGP is the maximum energy density that has been reached in the experiment. This maximum energy density is connected with the final state energy density and the phase space volume. For ultra high energy collisions, more than eighty percent of the final state particles are pions; therefore, it is very important to measure the phase space density of pions which can be used to infer the energy density in the early stage of the collisions [3]. Besides this, it is very important to estimate the phase space density also for the following reasons.

(1) If the boson phase space density is very large, then pions will tend to stay in the same state and a pion condensate may occur.

(2) If the density of pions is very large, then the mean free-length among pions will be small; therefore, the evolution of pions in the final state should be described by the hydrodynamical equation [4].

Bertsch suggested a method which uses pion interferometry measurements [5] to calculate the mean pion phase space density several years ago [6]. Since then several calculations have been done at AGS energy [7] and SPS energy [6, 8–12]. It has been found that the phase space density of pions is very low at AGS and SPS energies [6–8,10,12]; however this may be not the case at RHIC energies.

Bose–Einstein (BE) correlation effects on the pion multiplicity distribution, on the single pion distribution, and on two-pion interferometry have been studied by many authors for a Gaussian source distribution [13–17]. It has

been shown by Bialas and Zalewski that those results are valid for a wide class of models [18–20]. Fialkowski and Wit have implemented multi-particle Bose–Einstein correlations in Monte Carlo generators and have studied Bose–Einstein correlation effects on the W mass shift, the pion multiplicity distribution [21, 22]. But the effects of multi-particle Bose–Einstein correlations on the mean phase space density have never been studied before. Theoretical studies [16, 17] have shown that pion interferometry depends strongly on the pion multiplicity distribution which was overlooked in previous HBT analyses. Thus it is interesting for us to study the effects of the general pion interferometry formula on the mean phase space density.

This paper is arranged in the following way. In Sect. 2, we re-derive Bertsch's formula and point out its implicit assumption. In Sect. 3, we derive a phase space density formula from the general pion interferometry and we find that if the phase space density becomes small this new expression will be the same as the Bertsch's except a extra normalization factor. Unfortunately this simple relationship does not hold when the phase space density becomes large. In Sect. 4, multi-pion Bose–Einstein correlation effects on the mean phase space density are studied. We find that multi-particle BE correlations will increase the mean phase space density. Finally we give our conclusions in Sect. 5.

2 Bertsch's formula and implicit assumptions

The two-pion interferometry formula can be written as

$$C_2^{\mathrm{I}}(p_1, p_2) = \frac{P_2(p_1, p_2)}{P_1(p_1) P_1(p_2)} \tag{1}$$

$$= 1 + \frac{\int d^4x d^4y g^I(x,k) g^I(y,k) \exp(iq(x-y))}{\int d^4x d^4y g^I(x,p_1) g^I(x,p_2)}.$$

Here $k = (p_1 + p_2)/2$ and $q = p_1 - p_2$ are two-pions average momentum and relative momentum respectively. $g^I(x,k)$ is a Wigner function which can be interpreted as the probability of finding a pion at position x with momentum k. $P_2(p_1, p_2)$ and $P_1(p)$ are two-particle and single-particle inclusive distributions which are defined by

$$P_2(p_1, p_2) = \frac{d^6 n}{d^3 p_1 d^3 p_2}$$

$$= P_1(p_1) P_1(p_2)$$

$$+ \int g^I(x,k) g^I(y,k) \exp(iq(x-y)) d^4 x d^4 y,$$

$$P_1(p) = \frac{d^3 n}{d^3 p} = \int d^4 x g^I(x,p), \tag{2}$$

with

$$\int d^3 p P_1(p) = \langle n \rangle,$$

$$\int d^3 p_1 d^3 p_2 P_2(p_1, p_2) = \langle n(n-1) \rangle. \tag{3}$$

From (1) and (2), it is easily checked that if [6, 10]

$$g^I(x,k) \to \delta(x_0 - \tau) f^I(x,k) \frac{1}{(2\pi)^3}, \tag{4}$$

then

$$P_1(p) = \frac{1}{(2\pi)^3} \int f^I(x,p) d^3 x, \tag{5}$$

and

$$P_2(p_1, p_2) - P_1(p_1) P_1(p_2) \tag{6}$$

$$= \frac{1}{(2\pi)^6} \int d^3 x d^3 y f^I(x,k) f^I(y,k) \exp(iq(x-y)).$$

The reason that we put $\frac{1}{(2\pi)^3}$ in (4) is that in statistical physics for a infinite volume [23]

$$P_1(p) = \frac{V}{(2\pi)^3} f(p). \tag{7}$$

Here V is the volume and $f(p)$ is the Bose–Einstein distribution.

Integrating (6) over q we have

$$\int d^3 q [P_2(p_1, p_2) - P_1(p_1) P_1(p_2)]$$

$$= \frac{1}{(2\pi)^3} \int f^I(x,k)^2 d^3 x. \tag{8}$$

The average phase space density $\langle f^I \rangle_k$ can be defined as

$$\langle f^I \rangle_k = \frac{\int f^I(x,k)^2 d^3 x}{\int f^I(x,k) d^3 x}$$

$$= \frac{\int d^3 q \left[P_2 \left(k + \frac{q}{2}, k - \frac{q}{2} \right) - P_1 \left(k + \frac{q}{2} \right) P_1 \left(k - \frac{q}{2} \right) \right]}{P_1(k)}$$

$$= \frac{\int d^3 q [C_2^I(q,k) - 1] P_1 \left(k + \frac{q}{2} \right) P_1 \left(k - \frac{q}{2} \right)}{P_1(k)}. \tag{9}$$

Using the smooth approximation, $p_1 \sim p_2 \sim k$, which has been shown to be valid in heavy-ion collisions for its large phase space [24][1], we have

$$\langle f^I \rangle_k = P_1(k) \int d^3 q [C_2^I(k,q) - 1]. \tag{10}$$

In [7, 8, 10], the authors have calculated the phase space density by assuming that

$$C_2^I(p_1, p_2) \tag{11}$$

$$= 1 + \lambda \exp \left(-\frac{1}{2} q_o^2 R_o^2 - \frac{1}{2} q_s^2 R_s^2 - \frac{1}{2} q_l^2 R_l^2 - 2 R_o R_l q_o q_l \right).$$

But the above parameterization of two-pion interferometry, (11), is not general, as in practice the two-pion correlation is fitted using the function [15, 16]

$$C_2^{ex}(p_1, p_2) = A C_2^I(p_1, p_2). \tag{12}$$

Here A is a normalization factor which exists in the two-pion interferometry formula [15, 16]. If we use $C_2^{ex}(q,k)$ to take the place of $C_2^I(q,k)$ in (10), the phase space density will be

$$\langle f \rangle_k^{ex} = (A - 1) P_1(k) + A \times \langle f^I \rangle_k. \tag{13}$$

Thus the phase space density will increase if A is larger than one or decrease if A is smaller than one. In the latter part of this paper we will show that this extra factor A, though it has been used in the data analyses, will not appear in the phase space density formula on the condition that the phase space is large and the density is small. This guarantees that the application of the Bertsch formula for heavy-ion collisions system is appropriate if the phase space density is small.

3 Phase space density from the general pion interferometry formula

It has been shown in [16, 17] that the single-particle spectrum, the two-particle spectrum and the two-pion inter-

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[1] If there are strong correlations between coordinate and momentum spaces, then this smooth approximation may be not valid. See for example [25].

ferometry formula read[2]

$$P_1(p) = \sum_{i=1}^{N_{max}} h_i G_i(p, p), \quad (14)$$

$$P_2(p_1, p_2) = \sum_{i=1}^{N_{max}-1} \sum_{j=1}^{N_{max}-i} h_{i+j}[G_i(p_1, p_1)G_i(p_2, p_2) + G_i(p_1, p_2)G_i(p_2, p_1)], \quad (15)$$

where the definitions of h_i and $G_i(p, q)$ can be found in [17]. N_{max} is the maximum multiplicity in the experiment. If $N_{max} = \infty$, we will obtain the formula of $P_2(p_1, p_2)$ and $P_1(p)$ given in [16,17].

The two-pion correlation function is [17]

$$C_2(p_1, p_2) = \frac{P_2(p_1, p_2)}{P_1(p_1)P_1(p_2)}$$
$$= C_2^{res}(p_1, p_2)[1 + R_2(p_1, p_2)], \quad (16)$$

with

$$C_2^{res}(p_1, p_2)$$
$$= \frac{\sum_i^{N_{max}-1} \sum_{j=1}^{N_{max}-i} h_{i+j}G_i(p_1, p_1)G_j(p_2, p_2)}{\sum_{i,j=1}^{N_{max}} h_i h_j G_i(p_1, p_1)G_j(p_2, p_2)}, \quad (17)$$

and

$$R_2(p_1, p_2)$$
$$= \frac{\sum_{i=1}^{N_{max}-1} \sum_{j=1}^{N_{max}-i} h_{i+j}G_i(p_1, p_2)G_j(p_2, p_1)}{\sum_{i=1}^{N_{max}-1} \sum_{j=1}^{N_{max}-i} h_{i+j}G_i(p_1, p_1)G_j(p_2, p_2)}. \quad (18)$$

In [17], we have shown that $R_2(k, q)|_{q=\infty} = 0$. So if the two-pion correlation function is expressed as (12), then

$$A = C_2^{res}(p_1, p_2), \quad (19)$$

which is a function of q and k. We can always define a Wigner function $S(x, k)$ which fulfills the following equation [14, 15, 17]:

$$\sum_{i=1}^{N_{max}} h_i G_i(p_1, p_2) = \int S(x, k) \exp(iqx) \mathrm{d}^4 x. \quad (20)$$

Thus,

$$P_1(p) = \sum_{i=1}^{N_{max}} h_i G_i(p, p) = \int S(x, p) \mathrm{d}^4 x. \quad (21)$$

It has been shown in [14,15,17] that for a special multiplicity distribution $p_n = \frac{\omega(n)}{\sum_{n=0}^{n=\infty} \omega(n)}$ or for a small phase space

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density and p_n being a Poisson form, we have $h_{i+j} = h_i h_j$. Then (15) and (16) (for $N_{max} = \infty$) change to [14,15,17]

$$P_2(p_1, p_2) = P_1(p_1)P_1(p_2) \quad (22)$$
$$+ \int S(x, k)S(y, k) \exp(iq(x - y)) \mathrm{d}x \mathrm{d}y,$$

and

$$C_2(p_1, p_2) \quad (23)$$
$$= 1 + \frac{\int S(x, k)S(y, k) \exp(iq(x - y)) \mathrm{d}x \mathrm{d}y}{\int \mathrm{d}x \mathrm{d}y S(x, p_1)S(y, P_2)}.$$

So we obtain (1). Thus all the derivations given in [6] are valid. This implies that the normalization factor A must be one and this can be shown from (17) under the condition that $N_{max} = \infty$ and $h_{i+j} = h_i h_j$. From (13), we have

$$\langle f \rangle_k^{ex} = \langle f^I \rangle_k. \quad (24)$$

This verifies the rightness of the application of Bertsch's formula for the case of small boson densities. However, the relationship $h_{i+j} = h_i h_j$ does not hold for all cases. For the following four kinds of multiplicity distributions:

$$p_n = \frac{\langle n \rangle^n}{n!} \exp(-\langle n \rangle)$$

(Poisson distribution),

$$p_n = \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{n+1}}$$

(Bose-Einstein distribution),

$$p_n = \frac{(n + k - 1)!}{n!(k - 1)!} \frac{(\langle n \rangle / k)^n}{\left(1 + \frac{\langle n \rangle}{k}\right)^{n+k}}$$

(negative binomial distribution),

$$p_n = \frac{1}{n\Gamma(k)} \left(\frac{kn}{\langle n \rangle}\right)^k \exp(-kn/\langle n \rangle)$$

(Gamma distribution), $\quad (25)$

we can prove [17] that if the phase space volume is large, $h_2/h_1^2 \sim h_3/(h_1 h_2) \sim 1$ for the Poisson distribution, and $h_2/h_1^2 = 2$ and $h_3/(h_1 h_2) = 3$ for the Bose–Einstein distribution.

If there are strong BE correlations among the bosons, then it is impossible to express the two-particle distribution $S(x, p_1; y, p_2)$ as $S(x, p_1)S(y, p_2)$. However, one can find a real function $S_i(x, k)$ which fulfills the following equation:

$$G_i(p_1, p_2) = \int S_i(x, k) \exp(iqx) \mathrm{d}^4 x. \quad (26)$$

Then we define $S(x, k; y, k)$ and $S(x, p_1; y, p_2)$ as

$$S(x, k; y, k) = \sum_{i=1}^{N_{max}-1} \sum_{j=1}^{N_{max}-i} h_{i+j}S_i(x, k)S_j(y, k), (27)$$

and

$$S(x, p_1; y, p_2)$$

$$= \sum_{i=1}^{N_{\max}-1} \sum_{j=1}^{N_{\max}-i} h_{i+j} S_i(x, p_1) S_j(y, p_2), \qquad (28)$$

which satisfies

$$\sum_{i=1}^{N_{\max}-1} \sum_{j=1}^{N_{\max}-i} h_{i+j} G_i(p_1, p_2) G_j(p_2, p_1)$$

$$= \int S(x, k; y, k) \exp(iq(x - y)) dx dy \qquad (29)$$

and

$$\sum_{i=1}^{N_{\max}-1} \sum_{j=1}^{N_{\max}-i} h_{i+j} G_i(p_1, p_1) G_j(p_2, p_2)$$

$$= \int S(x, p_1; y, p_2) dx dy. \qquad (30)$$

Because $\rho^*(p_1, p_2) = \rho(p_2, p_1)$, then $G_i^*(p, p) = G_i(p, p)$ and $G_i^*(p, q) = G_i(q, p)$. Thus $S(x, p_1; y, p_2)$ must be a real function which fulfills the requirement of the Wigner function. If $h_{i+j} = h_i h_j$ and $N_{\max} \to \infty$, from (28), (26) and (20), we have $S(x, p_1; y, p_2) = S(x, p_1) S(y, p_2)$; thus (21), (22) and (23) are obtained. So we can identify $S(x, p_1, y, p_2)$ as a two pion distribution function which represents the probability of two pions emitted from point x with momentum p_1 and from point y with momentum p_2 respectively. The two-particle spectrum distributions can be written as

$$P_2(p_1, p_2) = \int S(x, p_1; y, p_2) dx dy \qquad (31)$$

$$+ \int S(x, k; y, k) \exp[iq(x - y)] dx dy.$$

Then $R_2(p_1, p_2)$ reads

$$R_2(p_1, p_2) = \frac{\int S(x, k; y, k) \exp(iq(x - y)) dx dy}{\int S(x, p_1; y, p_2) dx dy}. \qquad (32)$$

If [6]

$$S(x, k; y, k) \to \delta(x_0 - \tau) \delta(y_0 - \tau) f(x, k; y, k) \frac{1}{(2\pi)^6} \qquad (33)$$

and

$$S(x, k) \to \delta(x_0 - \tau) \frac{1}{(2\pi)^3} f(x, k), \qquad (34)$$

then

$$\langle f \rangle_k = \frac{\int f(x, k; x, k) d^3 x}{\int f(x, k) d^3 x} \qquad (35)$$

$$= \frac{\int d^3 q R_2(p_1, p_2) C_2^{\text{res}}(p_1, p_2) P_1(p_1) P_1(p_2)}{P_1(k)}.$$

Using the smoothness approximation, $P_1(p_1) \sim P_1(p_2) \sim P_1(k)$, we have

$$\langle f \rangle_k = P_1(k) \int d^3 q \left[\frac{C_2(p_1, p_2)}{C_2^{\text{res}}(p_1, p_2)} - 1 \right] C_2^{\text{res}}(p_1, p_2)$$

$$= A P_1(k) \int d^3 q \left[\frac{C_2(q, k)}{A} - 1 \right]$$

$$= A \langle f^{\text{I}} \rangle_k. \qquad (36)$$

In the above we have used the approximation $A \sim C_2^{\text{res}}(q, k) \sim$ const. It is interesting to point out that normally A should be a function of q and k; however, in heavy-ion collisions, the practice is to normally fit it as a constant if the phase space density is small [17]. If the phase space density is high, as has been suggested in [17], to fit the data one might use the function

$$C_2(p_1, p_2) = C_2^{\text{res}}(p_1, p_2)[1 + R_2(p_1, p_2)]. \qquad (37)$$

Here $C_2^{\text{res}}(q, k) = \mathcal{N}[1 + B(k) \cdot \exp(-q^2 R_{\text{res}}^2(k))]$ and $R_2 = \lambda(k) \exp(-q^2 R^2(k))$. Thus it is not the best choice to use a constant A in (12).

4 Multi-particle BE correlation effects on the mean phase space density

In the following we will study the effects of multi-pion BE correlations on the mean phase space density. We assume $g^{\text{I}}(x, p)$ to be [13, 26]

$$g^{\text{I}}(x, p) = \delta(x_0) \frac{n_0}{(2\pi R \Delta)^3} \exp\left(-\frac{x^2}{2R^2} - \frac{p^2}{2\Delta^2}\right); \qquad (38)$$

then $f^{\text{I}}(x, p)$ reads

$$f^{\text{I}}(x, p) = \frac{(2\pi)^3 n_0}{(2\pi R \Delta)^3} \exp\left(-\frac{x^2}{2R^2} - \frac{p^2}{2\Delta^2}\right). \qquad (39)$$

Due to (5), we immediately come to the conclusion that n_0 is the mean pion multiplicity observed in the experiment. It is easily checked that

$$\langle f^{\text{I}} \rangle_k = \frac{\int d^3 x f^{\text{I}}(x, p)^2}{\int d^3 x f^{\text{I}}(x, p)} = \frac{n_0}{(\sqrt{2} R \Delta)^3} \exp\left(-\frac{p^2}{2\Delta^2}\right). \qquad (40)$$

However, we have neglected the high-order BE correlation effects to get this $\langle f^{\text{I}} \rangle_k$. If we keep only the leading terms in (19) and (20) (correspondingly, this implies that we have assumed that the phase space volume is large), then

$$P_1(p) = h_1 G_1(p, p), \qquad (41)$$

and

$$P_2(p_1, p_2) \qquad (42)$$

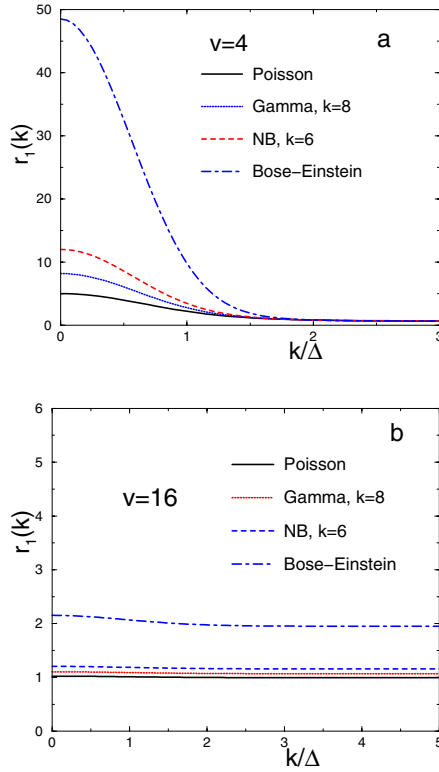$$= h_2[G_1(p_1, p_1) G_1(p_2, p_2) + G_1(p_1, p_2) G_1(p_2, p_1)].$$

Fig. 1a,b. $r_1(k)$ as a function of k/Δ. Here the mean pion multiplicity is 20. The solid line, dashed line, dotted line and dot-dashed lines correspond to Poisson, Gamma, negative binomial, and Bose–Einstein distribution. The phase space is $v = 2R\Delta = 4$ and 16 respectively

Thus the corresponding two-pion correlation function becomes

$$C_2(p_1, p_2) = \frac{h_2}{h_1^2}\left[1 + \frac{G_1(p_1, p_2)G_1(p_2, p_1)}{G_1(p_1, p_1)G_1(p_2, p_2)}\right]. \quad (43)$$

Bringing (43) into (9) and ignoring the extra normalization factor, $\frac{h_2}{h_1^2}$, we get (40); furthermore, if we bring (43) into (35) and ignore the extra normalization factor, (40) will be regained. These results demonstrate that the Bertsch formula will give the correct result on the condition that the phase space volume is large. In the following we will study the multi-particles correlation effects on the mean phase space density due to the large phase space density.

We define the function $r_1(k)$ by

$$r_1(k) = \frac{\langle f \rangle_k}{\langle f^I \rangle_k}. \quad (44)$$

In Fig.1, $r_1(k)$ is shown as a function of k/Δ. It is clear that for a large phase space, $r_1(k) \sim$ constant; on the other hand, if the phase space is small, $\langle f \rangle_k$ and $\langle f^I \rangle_k$ have big differences at small momentum and small differences at large momentum. This is easily understood since quantum effects are big for small momentum particles. It is interesting to notice that when the phase space is large, $r_1 \sim h_2/h_1^2 (\sim C^{\mathrm{res}}(q, k))$, which is two for the Bose–Einstein distribution and one for the Poisson distribution.
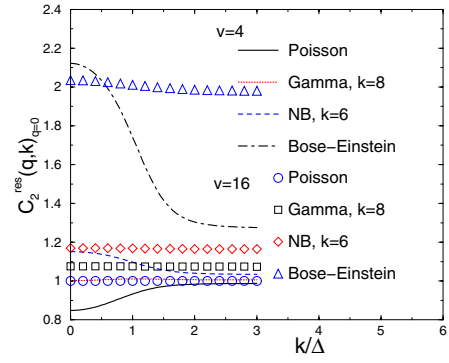

Fig. 2. $C_2^{\mathrm{res}}(q, k)|_{q=0}$ versus k/Δ. Here the mean pion multiplicity is 20. The solid line (circles), dashed line (squares), dotted line (diamonds) and dot-dashed lines (triangles) corresponds to Poisson, Gamma, negative binomial, and Bose–Einstein distribution. The phase space is $v = 2R\Delta = 4$ and 16 respectively

From (33) and (29) (taking $q = 0$), we have

$$\int f(x, k; y, k)\mathrm{d}^3x\mathrm{d}^3y$$

$$= (2\pi)^6 \int S(x, k; y, k)\mathrm{d}^4x\mathrm{d}^4y$$

$$= (2\pi)^6 \sum_{i=1}^{N_{\max}-1}\sum_{j=1}^{N_{\max}-i} h_{i+j}G_i(k, k)G_j(k, k). \quad (45)$$

From (34) and (20) (taking $q = 0$), we get

$$\int f(x, k)\mathrm{d}x = (2\pi)^3 \int S(x, k)\mathrm{d}^4x$$

$$= (2\pi)^3 \sum_{i=1}^{N_{\max}} h_iG_i(k, k). \quad (46)$$

From the definition of $C_2^{\mathrm{res}}(p_1, p_2)$((17)) and (45) and (46), we have

$$C_2^{\mathrm{res}}(q, k)_{q=0} = \frac{\int f(x, k; y, k)\mathrm{d}^3x\mathrm{d}^3y}{\int \mathrm{d}^3x\mathrm{d}^3y f(x, k)f(y, k)}. \quad (47)$$

In Fig. 2, $C_2^{\mathrm{res}}(q, k)_{q=0}$ versus k/Δ is shown. We notice that when phase space is large, $C_2^{\mathrm{res}}(q, k)_{q=0}$ is a constant. In this case,

$$\int \mathrm{d}^3x\mathrm{d}^3y f(x, k; y, k) = 2\int \mathrm{d}^3x\mathrm{d}^3y f(x, k)f(y, k)$$

for the Bose–Einstein distribution and

$$\int \mathrm{d}^3x\mathrm{d}^3y f(x, k; y, k) = \int \mathrm{d}^3x\mathrm{d}^3y f(x, k)f(y, k)$$

for the Poisson multiplicity distribution. But these relationships do not hold anymore for a small phase space volume.

In (9) and (36), we find that $\langle f \rangle_k$ is A times larger than $\langle f^I \rangle_k$ when the phase space density is small and the function forms of $\langle f \rangle_k$ and $\langle f^I \rangle_k$ are different in the numerator. In the following we would like to show the relationship between $f(x, k; x, k)$ and $f(x, k)$. It is found that

$$\int f(x, k; x, k) \mathrm{d}^3 x$$

$$= \frac{1}{(2\pi)^3} \int f(x, k; y, k) e^{\mathrm{i}\mathbf{q}(\mathbf{x}-\mathbf{y})} \mathrm{d}^3 q \mathrm{d}^3 x \mathrm{d}^3 y$$

$$= (2\pi)^3 \int S(x, k; y, k) e^{\mathrm{i}q(x-y)} \mathrm{d}^4 x \mathrm{d}^4 y \mathrm{d}^3 q \qquad (48)$$

$$= (2\pi)^3 \int \mathrm{d}^3 q \sum_{i=1}^{N_{\max}-1} \sum_{j=1}^{N_{\max}-i} h_{i+j} G_i(p_1, p_2) G_j(p_2, p_1)$$

and

$$\int f^2(x, k) \mathrm{d}^3 x \qquad (49)$$

$$= (2\pi)^3 \int \mathrm{d}^3 q \sum_{i=1}^{N_{\max}} \sum_{j=1}^{N_{\max}} h_i h_j G_i(p_1, p_2) G_j(p_2, p_1).$$

In the above derivation, we have used (20), (26), (29), (33) and (34). We define the function $r_2(k)$ by

$$r_2(k) = \frac{\int f(x, k; x, k) \mathrm{d}^3 x}{\int \mathrm{d}^3 x f^2(x, k)}. \qquad (50)$$

Similar to [17], we can prove that

$$r_2(k) \sim C_2^{\mathrm{res}}(q, k) \sim \frac{h_2}{h_1^2} = \text{constant}, \quad v \to \infty . \qquad (51)$$

In the following, we will discuss the effects of multi-boson correlations on the distribution function of pions in momentum space. If the multi-boson symmetrization effects are small, the distribution function is $f^I(x, p)$. If the multi-boson correlations are strong, the distribution function is $f(x, k)$. We define the function

$$r_3(k) = \frac{\int \mathrm{d}^3 x f(x, k)}{\int f^I(x, k) \mathrm{d}^3 x} = \frac{(2\pi)^3 P_1(k)}{\int f^I(x, k) \mathrm{d}^3 x}, \qquad (52)$$

which reflects the effects of multi-boson correlations on the distribution function. In Fig. 3, $r_3(k)$ versus k/Δ is shown. It is interesting to notice that when phase space is large,

$$r_3(k) = 1, v \to \infty \qquad (53)$$

for all distributions. On the other hand, when phase space is small, the boson density becomes large at small momentum region. We define $\langle f \rangle_k^I$ by

$$\langle f \rangle_k^I = \frac{\int f^2(x, k) \mathrm{d}x}{\int f(x, k) \mathrm{d}x}. \qquad (54)$$
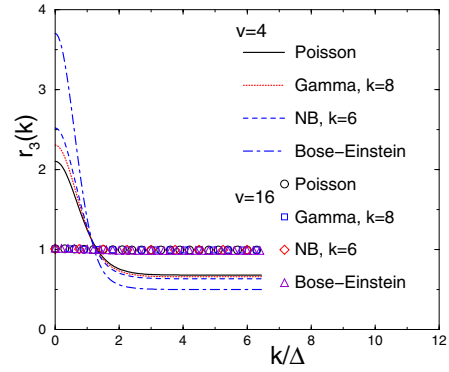

Fig. 3. $r_3(k)$ as a function of k/Δ. Here the mean pion multiplicity is 20. The solid line (circles), dashed line (squares), dotted line (diamonds) and dot-dashed lines (triangles) corresponds to Poisson, Gamma, negative binomial, and Bose–Einstein distribution. The phase space is $v = 2R\Delta = 4$ and 16 respectively

This definition is similar to (9) but with $f(x, k)$ taking the place of $f^I(x, k)$. From (9), (12), (35), (36) and (51), we get

$$\langle f \rangle_k^I = \frac{\langle f \rangle_k}{r_2(k)} \qquad (55)$$

$$\sim P(k) \int \mathrm{d}^3 q \left[\frac{C_2(q, K)}{A} - 1 \right] \quad v \to \infty.$$

This is one of the main results of this paper. This result guarantees that the application of Bertsch's formula for a large system and dilute gas is appropriate. On the other hand, Bertsch formula is incomplete when the phase space density is high. This comes from the fact that we cannot calculate $\int f^2(x, k) \mathrm{d}^3 x$ from the two-pion interferometry formula though we could calculate $\int f(x, k) \mathrm{d}^3 x$ from the single-particle spectrum. If the phase space density is high, $C_2^{\mathrm{res}}(p_1, p_2)$ is not a constant anymore, we cannot find the approximation formula as (55); however, (35) can still be used to find the ratio of the number of particles pairs which are emitted from the same phase space cell to the average number of particles. The relation among $\langle f \rangle_k$, $\langle f \rangle_k^I$ and $\langle f^I \rangle_k$ will be very complex when the phase space density is large. Then the physical meaning of the Bertsch formula is no longer clear. Multi-pion BE symmetrization will affect the current formalism and we believe that those effects will be similar to the effects of a multi-pion BE on the Wigner function $g(x, p)$, which have been presented in [15] (or $f^I(x, p)$ presented here).

5 Conclusions

In this paper, the mean phase space density distribution of bosons is derived from the general pion interferometry formula. We find that when the phase space is small and thus the boson density is high, the two particle source distribution cannot be expressed as a product of two single-particle source distributions. On the other hand, when the phase

space is large and thus the boson density is small, Bertsch's formula is recovered. Thus Bertsch's formula can be used for the heavy-ion system if the freeze-out pion phase space density is small. Multi-pion BE correlation effects on the mean phase space density distribution are studied, and it is found that when the phase space density is large, bosons are concentrated in a small momentum region and this effect is connected with the pion multiplicity distributions.

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